

# Solving a sparse systems using linear algebra.

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## Abstract

We generalize a method to compute the solutions of a system of polynomial equations with finitely many solutions. We solve this problem without the need of Gröbner bases.

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## Introduction.

In this paper we adapt a method, presented in [4, Ch. 3, §6], that solves a system with  $n$  polynomial equations and  $n$  variables with finitely many solutions, to a method that solves a general sparse system with finitely many solutions.

In **Section 1** we present the general method to solve a sparse system with finitely many solutions. In **Section 2** and **Section 3** we adapt the method to solve a system of trilinear equations and a system of polynomial equations respectively.

The method is based on the following theorems. Consider a system of polynomial equations with finitely many solutions in  $\mathbb{C}^n$ ,

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_k(x_1, \dots, x_n) = 0 \end{cases}$$

where  $f_1, \dots, f_k$  are polynomials in  $\mathbb{C}[x_1, \dots, x_n]$ . The quotient ring,

$$\mathcal{R} = \mathbb{C}[x_1, \dots, x_n]/\langle f_1, \dots, f_k \rangle,$$

is a finite-dimensional vector space, [6, Theorem 2.1.2]. The dimension of  $\mathcal{R}$  is the number of solutions of the system counted with multiplicities.

Every polynomial  $f \in \mathbb{C}[x_1, \dots, x_n]$ , determines a linear map  $M : \mathcal{R} \rightarrow \mathcal{R}$ ,

$$M(\overline{g}) = \overline{fg}, \quad g \in \mathbb{C}[x_1, \dots, x_n],$$

where  $\overline{g}$  denotes the class of the polynomial  $g$  in the quotient ring  $\mathcal{R}$ . The matrix of  $M$  is called the *multiplication matrix* assigned to the polynomial  $f$ .

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**Theorem** (Eigenvalue Theorem). *The eigenvalues of  $M$  are  $\{f(\xi_1), \dots, f(\xi_r)\}$ , where  $\{\xi_1, \dots, \xi_r\}$  are the solutions of the system of polynomial equations. See [6, Theorem 2.1.4] for a proof.*

**Theorem** (Eigenvector Theorem). *Let  $f = \alpha_1 x_1 + \dots + \alpha_n x_n$  be a generic linear form and let  $M$  be its multiplication matrix. Assume that  $B = \{1, x_1, \dots, x_n, \dots\}$  is a finite basis of  $\mathcal{R}$  formed by monomials. Then the eigenvectors of  $M$  determine all the solutions of the system of polynomial equations. Specifically, if  $v = (v_0, \dots, v_n, \dots)$  is an eigenvector of  $M$  such that  $v_0 = 1$ , then  $(v_1, \dots, v_n)$  is a solution of the system of polynomial equations. Even more, every solution is of this form. See [6, §2.1.3] for a proof.*

Note that the previous theorem requires that the variables  $\{x_1, \dots, x_n\}$  are elements of the basis  $B$ . It could be the case that some variables are missing from  $B$ . For example, if  $x_1, \dots, x_i \in B$ , and  $x_{i+1}, \dots, x_n \notin B$ , then every missing variable, say  $x_j$ , is a linear combination of  $\{x_1, \dots, x_i\}$ ,

$$x_j = a_{j1}x_1 + \dots + a_{ji}x_i, \quad i+1 \leq j \leq n.$$

If  $v = (1, v_1, v_2, \dots)$  is an eigenvector of  $M$ , the  $j$ -coordinate of the solution corresponding to  $v$ , is  $a_{j1}v_1 + \dots + a_{ji}v_i$ . See [6, §2.1.3].

The computation of the multiplication matrix is a very difficult task to do. In general, we need Gröbner bases. When the system of polynomial equations has the same number of variables and equations,  $n = k$ , there exists a method to compute the multiplication matrix, see [2] and [6, §2.3.1].

In [4, §7.6.21], the authors adapted the method to solve a sparse system with the same number of variables and equations. Here, we adapt their method to solve a general sparse system with finitely many solutions. We needed to prove Theorem 1 in order to generalize it. The proof is based on standard algebro-geometric tools and Toric varieties theory.

## 1. Solving a sparse system.

A *sparse system* is a collection of Laurent polynomials,

$$f_i = \sum_{v \in Q_i} c_{i,v} x_1^{v_1} \dots x_n^{v_n}, \quad 1 \leq i \leq k,$$

where  $Q_i$  are fixed finite subsets of  $\mathbb{Z}^n$ . We say that the system is *generic* if  $k = n$  and the coefficients  $c_{i,v}$  are chosen arbitrarily. The set  $Q_i$  is called the *support* of  $f_i$ .

Bernstein proved in [3] that the number of solutions in  $(\mathbb{C} \setminus \{0\})^n$  of a generic sparse system equals the *mixed volume* of the corresponding *Newton polytopes*. The convex hull  $\mathcal{A}_i$  of  $Q_i$ ,

$$\mathcal{A}_i = \text{conv}(Q_i) \subseteq \mathbb{R}^n,$$

is called the Newton polytope of  $f_i$ , denoted  $N(f_i)$ .

Consider the function

$$V(\lambda_1, \dots, \lambda_n) := \text{vol}(\lambda_1 \mathcal{A}_1 + \dots + \lambda_n \mathcal{A}_n), \quad \lambda_i \geq 0, 1 \leq i \leq n.$$

where *vol* is the usual Euclidean volume in  $\mathbb{R}^n$  and  $\mathcal{A} + \mathcal{A}'$  denotes the Minkowski sum of polytopes. It is a fact that  $V$  is a homogeneous polynomial and the coefficient of the monomial

$\lambda_1 \dots \lambda_n$  is called the mixed volume of  $\mathcal{A}_1, \dots, \mathcal{A}_n$ . The mixed volume (i.e the number of solutions of a generic sparse system) is a very difficult number to compute, see [4, p. 363]. In some situations, this is possible and in the general case, there are a lot of algorithms to compute it. When the polynomials  $f_i$  are multihomogeneous, in the paper [11], the author gave a recursive formula to compute this number. When the system is homogeneous, we recover Bézout theorem.

Consider a sparse system with finitely many solutions,

$$\begin{cases} f_1 = \sum_{v \in Q_1} c_{1,v} x_1^{v_1} \dots x_n^{v_n} = 0 \\ \vdots \\ f_k = \sum_{v \in Q_k} c_{k,v} x_1^{v_1} \dots x_n^{v_n} = 0 \end{cases}$$

If the system is generic, the number of equations coincide with the number of variables,  $k = n$ , and the number of solutions is given by the mixed volume of the Newton polytopes of  $f_1, \dots, f_k$ . For example, the following sparse system is generic with only one solution,  $\xi = (1, 1)$ ,

$$\begin{cases} 1 - x_1 x_2 = 0 \\ 1 - x_2 = 0 \end{cases}$$

The supports, in  $\mathbb{Z}^2$ , of the equations are

$$Q_1 = \{(0, 0), (1, 1)\}, \quad Q_2 = \{(0, 0), (0, 1)\}.$$

Note that the Bézout number of this system is 2 and its mixed volume is 1.

Given a sparse system, it is easy to give a bound for the number of solutions. Multiply each equation by a positive power of a variable, and consider the corresponding Bézout number of the resulting system. It is always a bound for the number of solutions. If the system is a generic system of homogeneous polynomials, we can use Bézout's theorem to know the number of solutions,  $r = \deg(f_1) \dots \deg(f_n)$ .

Let  $S = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  be the algebra of Laurent polynomials. Given a Newton polytope,  $\mathcal{A}$ , let  $S_{\mathcal{A}}$  be the vector space of polynomials with Newton polytope in  $\mathcal{A}$ ,

$$S_{\mathcal{A}} = \{g \in S : N(g) \subseteq \mathcal{A}\}.$$

The dimension of  $S_{\mathcal{A}}$  is equal to the cardinal of  $\mathcal{A} \cap \mathbb{Z}^n$ ,

$$\dim(S_{\mathcal{A}}) = \#(\mathcal{A} \cap \mathbb{Z}^n).$$

The finite set  $\mathcal{A} \cap \mathbb{Z}^n$  determines a monomial basis for  $S_{\mathcal{A}}$ .

Let  $\mathcal{A}'$  be an arbitrary Newton polytope. Using the equality,

$$N(g_1 g_2) = N(g_1) + N(g_2), \quad g_1, g_2 \in S,$$

we obtain that the multiplication map is surjective,

$$\begin{aligned} \mu : S_{\mathcal{A}} \otimes S_{\mathcal{A}'} &\rightarrow S_{\mathcal{A} + \mathcal{A}'}, \\ \mu(x_1^{v_1} \dots x_n^{v_n} \otimes x_1^{v'_1} \dots x_n^{v'_n}) &= x_1^{v_1 + v'_1} \dots x_n^{v_n + v'_n}, \quad \forall v \in \mathcal{A}, v' \in \mathcal{A}'. \end{aligned}$$

In other words, its image,  $S_{\mathcal{A}} \cdot S_{\mathcal{A}'}$ , is equal to  $S_{\mathcal{A} + \mathcal{A}'}$ ,

$$S_{\mathcal{A}} \cdot S_{\mathcal{A}'} = S_{\mathcal{A} + \mathcal{A}'},$$

**Definition.** [5, Definition 2.2.9]. A Newton polytope  $\mathcal{A}$  is called *normal* if

$$(k \cdot \mathcal{A}) \cap \mathbb{Z}^n = (\mathcal{A} \cap \mathbb{Z}^n) + \dots + (\mathcal{A} \cap \mathbb{Z}^n), \quad \forall k \in \mathbb{N}.$$

The expression in the right has  $k$  terms.

Recall from [5, Theorem 2.2.12] that given a full dimensional Newton polytope  $\mathcal{A}$ , the Newton polytope  $(n-1) \cdot \mathcal{A}$  is normal. In particular, given  $\mathcal{A}$ , an arbitrary Newton polytope,  $(n-1) \cdot \mathcal{A}$  is always normal.

**Notation.** The following notations and assumptions will be used in the rest of the section.

Let  $\mathcal{J} \subseteq S$  be the ideal generated by a sparse system  $f_1, \dots, f_k$ , where the Newton polytope of  $f_i$  is  $\mathcal{A}_i$ ,  $1 \leq i \leq k$ .

$$\mathcal{J} = \langle f_1, \dots, f_k \rangle \subseteq S.$$

Assume that the sparse system has finitely many solutions, say  $r$ , and that the dimension of  $S/\mathcal{J}$  is also  $r$ . In other words, all the solutions have multiplicity one.

Let  $f_0$  be a non-constant generic Laurent polynomial and consider the ideal  $\mathcal{I}$ ,

$$\mathcal{I} = \langle f_0, f_1, \dots, f_k \rangle, \quad f_0 = \sum_{v \in \mathcal{A}_0 \cap \mathbb{Z}^n} c_{0,v} x_1^{v_1} \dots x_n^{v_n}, \quad N(f_0) = \mathcal{A}_0.$$

Without loss of generality, we may suppose that  $0 \in \mathcal{A}_i$  for every  $i$ ,  $0 \leq i \leq k$ . Divide each equation by a monomial to guarantee  $0 \in \mathcal{A}_i$ ,  $0 \leq i \leq k$ . This operation does not change the solutions. Recall that the solutions are in  $(\mathbb{C} \setminus 0)^n$ .

Assume also, that the Newton polytopes  $\mathcal{A}_0, \dots, \mathcal{A}_k$  are normal.

**Theorem 1.** Let  $\mathcal{E} = \mathcal{A}_0 + \dots + \mathcal{A}_k$  and let  $\mathcal{B}_i = \mathcal{A}_0 + \dots + \widehat{\mathcal{A}_i} + \dots + \mathcal{A}_k$ ,  $0 \leq i \leq k$ . Then the cokernel of the following linear map is isomorphic to  $S/\mathcal{J}$ ,

$$\Phi : S_{\mathcal{B}_1} \times \dots \times S_{\mathcal{B}_k} \rightarrow S_{\mathcal{E}}, \quad \Phi(g_1, \dots, g_k) = \sum_{i=1}^k f_i \cdot g_i,$$

The Newton polytope  $\mathcal{B}_0 \subseteq \mathcal{E}$  satisfies

$$S_{\mathcal{B}_0}/(im\Phi \cap S_{\mathcal{B}_0}) \cong S_{\mathcal{E}}/im\Phi.$$

Even more, the following linear map is surjective,

$$\Psi : S_{\mathcal{B}_0} \times S_{\mathcal{B}_1} \times \dots \times S_{\mathcal{B}_k} \longrightarrow S_{\mathcal{E}}, \quad \Psi(g_0, g_1, \dots, g_k) = f_0 \cdot g_0 + \sum_{i=1}^k f_i \cdot g_i.$$

**PROOF.** The basic references are [5] and [9].

Given a very ample Newton polytope  $\mathcal{A}$ , [5, Definition 2.2.17], we can construct a projective toric variety  $X$  and an invertible sheaf  $\mathcal{O}_X(1)$ , see [9, §8 1A]. Let  $X$  be the closure of the image of the following map,

$$(\mathbb{C} \setminus 0)^n \rightarrow \mathbb{P}(S_{\mathcal{A}}^\vee), \quad (t_1, \dots, t_n) \mapsto (t^{v_1} : \dots : t^{v_N}),$$

where  $t^w = t_1^{w_1} \dots t_n^{w_n}$  and  $\mathcal{A} \cap \mathbb{Z}^n = \{v_1, \dots, v_N\}$ . The invertible sheaf  $\mathcal{O}_X(1)$  is the restriction of  $\mathcal{O}_{\mathbb{P}(S_{\mathcal{A}}^\vee)}(1)$  to  $X$ . Recall from [5, Proposition 2.2.18] that a normal Newton polytope is very ample.

Let  $X_i \subseteq \mathbb{P}(S_{\mathcal{A}_i}^\vee)$  be the projective toric variety assigned to the polytope  $\mathcal{A}_i$ ,  $0 \leq i \leq k$ , [9, §5, 1B]. Consider the following invertible sheaves in  $X = X_0 \times \dots \times X_k$ ,

$$\mathcal{O}_X(d_0, \dots, d_k) = \pi_0^* \mathcal{O}_{X_0}(d_0) \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} \pi_k^* \mathcal{O}_{X_k}(d_k), \quad d_0, \dots, d_k \in \mathbb{Z},$$

where the map  $\pi_i : X \rightarrow X_i$  is the  $i$ -projection,  $0 \leq i \leq k$ .

By [9, Theorem 5.4.8], the coordinate ring of  $X_i$  is

$$\mathbb{C}[X_i] = \bigoplus_{d \geq 0} S_{d \cdot \mathcal{A}_i}, \quad 0 \leq i \leq k.$$

This implies that the space of global sections of  $\mathcal{O}_X(d_0, \dots, d_k)$  is,

$$H^0(X, \mathcal{O}_X(d_0, \dots, d_k)) = S_{d_0 \mathcal{A}_0} \otimes \dots \otimes S_{d_k \mathcal{A}_k}, \quad d_0, \dots, d_k \geq 0.$$

Let us work with the projective toric variety  $Y$  inside  $X$  given by,

$$\begin{array}{ccc} (t, \dots, t) & (\mathbb{C} \setminus 0)^n \times \dots \times (\mathbb{C} \setminus 0)^n & \longrightarrow X = X_0 \times \dots \times X_k \\ \uparrow t & \uparrow & \uparrow \\ (\mathbb{C} \setminus 0)^n & \longrightarrow & Y \end{array}$$

where the bottom map is the one induced by the Newton polytope  $\mathcal{E} = \mathcal{A}_0 + \dots + \mathcal{A}_k$  and the diagram is commutative, [9, §8, 1A, Proposition 1.4]. Let  $\mathcal{O}_Y(d_0, \dots, d_k)$  be the restriction of  $\mathcal{O}_X(d_0, \dots, d_k)$  to  $Y$ ,

$$\mathcal{O}_Y(d_0, \dots, d_k) = \mathcal{O}_X(d_0, \dots, d_k)|_Y.$$

A local affine section of  $\mathcal{O}_X(d_0, \dots, d_k)$  is generated by sections of the form  $t_0^{v_0} \cdots t_k^{v_k}$ , where  $v_i \in d_i \cdot \mathcal{A}_i$  and  $t_i = (t_{i1}, \dots, t_{in})$ ,  $0 \leq i \leq k$ . Restricting to  $Y$ , we get sections  $t^{v_0+...+v_k}$ . Then

$$H^0(Y, \mathcal{O}_Y(d_0, \dots, d_k)) = S_{d_0 \mathcal{A}_0 + \dots + d_k \mathcal{A}_k}, \quad d_0, \dots, d_k \geq 0.$$

Let  $e_i \in \mathbb{Z}^{k+1}$  be the vector with 1 in the  $(i+1)$ -coordinate and 0 in the rest,  $0 \leq i \leq k$ . For example,  $e_0 = (1, 0, \dots, 0)$  and  $e_k = (0, \dots, 0, 1)$ . The Laurent polynomials  $\{f_1, \dots, f_k\}$  determine a  $\mathcal{O}_Y$ -linear map

$$\mathcal{O}_Y \rightarrow \mathcal{F}, \quad \mathcal{F} = \mathcal{O}_Y(e_1) \oplus \dots \oplus \mathcal{O}_Y(e_k),$$

and then, we can construct the dual Koszul complex assigned to  $\mathcal{F}$ ,

$$0 \rightarrow \bigwedge^k \mathcal{F}^\vee \rightarrow \dots \rightarrow \bigwedge^i \mathcal{F}^\vee \rightarrow \dots \rightarrow \mathcal{F}^\vee \rightarrow \mathcal{O}_Y,$$

where

$$\bigwedge^i \mathcal{F}^\vee = \bigoplus_{1 \leq i_1 \leq \dots \leq i_s \leq k} \mathcal{O}_Y(-e_{i_1} - \dots - e_{i_s}).$$

Let  $Z \subseteq (\mathbb{C} \setminus 0)^n$  be the zero locus of  $\{f_1, \dots, f_k\}$ . Given that  $(\mathbb{C} \setminus 0)^n$  is embedded in  $Y$ , the variety  $Z$  may also be embedded in  $Y$ ,

$$Z \hookrightarrow Y, \quad \xi \rightarrow (\xi^{v_1} : \dots : \xi^{v_N}), \quad \{v_1, \dots, v_N\} = \mathcal{E} \cap \mathbb{Z}^n.$$

The dual Koszul complex assigned to  $\mathcal{F}$ , is supported in  $Z$ , [9, §B, Proposition 1.4 (a)]. Then the augmented complex is exact,

$$0 \rightarrow \bigwedge^k \mathcal{F}^\vee \rightarrow \dots \rightarrow \bigwedge^i \mathcal{F}^\vee \rightarrow \dots \rightarrow \mathcal{F}^\vee \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Z \rightarrow 0.$$

The exactness is preserved by twisting with the invertible sheaf  $\mathcal{O}_Y(1, \dots, 1)$ , and taking global sections. In particular, the following complex of vector spaces is exact,

$$S_{\mathcal{B}_1} \times \dots \times S_{\mathcal{B}_k} \xrightarrow{\Phi} S_{\mathcal{E}} \rightarrow S/\mathcal{J} \rightarrow 0.$$

Twisting the dual Koszul complex assigned to  $\mathcal{F}$  with the invertible sheaf  $\mathcal{O}_Y(0, 1, \dots, 1)$ , we obtain, in a similar way, that the following map is surjective,

$$S_{\mathcal{B}_0} \rightarrow S/\mathcal{J} \rightarrow 0.$$

Note that the kernel of  $S_{\mathcal{B}_0} \rightarrow S/\mathcal{J}$  is equal to  $S_{\mathcal{B}_0} \cap \mathcal{J}$ , but given that  $0 \in \mathcal{A}_0 \cap \dots \cap \mathcal{A}_k$ , we get  $\mathcal{B}_0 \subseteq \mathcal{E}$ , and then

$$S_{\mathcal{B}_0} \cap \mathcal{J} = S_{\mathcal{B}_0} \cap S_{\mathcal{E}} \cap \mathcal{J} = S_{\mathcal{B}_0} \cap \text{im } \Phi \implies S_{\mathcal{B}_0}/(\text{im } \Phi) \cong S/\mathcal{J} \cong S_{\mathcal{E}}/\text{im } \Phi.$$

Finally, if the Laurent polynomial  $f_0$  is non-zero over  $Z$ , that is, if the ideal  $\mathcal{I}$  is equal to  $S$ , then similar arguments, using the sheaf  $\mathcal{F}' = \mathcal{O}_Y(e_0) \oplus \dots \oplus \mathcal{O}_Y(e_k)$ , show that the following map is surjective,

$$S_{\mathcal{B}_0} \times \dots \times S_{\mathcal{B}_k} \xrightarrow{\Psi} S_{\mathcal{E}} \rightarrow 0.$$

□

Consider  $\mathcal{E} \cap \mathbb{Z}^n$  and  $\mathcal{B}_i \cap \mathbb{Z}^n$ ,  $0 \leq i \leq k$ , as monomial ordered bases of  $S_{\mathcal{E}}$  and  $S_{\mathcal{B}_i}$ ,  $0 \leq i \leq k$ , respectively. Let  $p$  be the cardinal of  $\mathcal{B}_0 \cap \mathbb{Z}^n$ ,  $p_i$  the cardinal of  $\mathcal{B}_i \cap \mathbb{Z}^n$ ,  $1 \leq i \leq k$  and  $p+q$  be the cardinal of  $\mathcal{E} \cap \mathbb{Z}^n$ . Given that  $f_0$  is non-constant, the inclusion,  $\mathcal{B}_0 \subseteq \mathcal{E}$ , is proper, thus  $q > 0$ .

$$\mathcal{B}_0 \cap \mathbb{Z}^n = \{m_1, \dots, m_p\}, \quad \mathcal{E} \cap \mathbb{Z}^n = \{m_1, \dots, m_p, m_{p+1}, \dots, m_{p+q}\},$$

where  $m_i$  is a monomial,  $1 \leq i \leq p+q$ . Here we are abusing the notation. The monomial  $m = x_1^{v_1} \dots x_n^{v_n}$  corresponds to the point  $(v_1, \dots, v_n) \in \mathbb{Z}^n$ .

Let  $M \in \mathbb{C}^{p+q \times p+p_1+\dots+p_k}$  be the rectangular matrix assigned to  $\Psi$  in these bases,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad M_{11} \in \mathbb{C}^{p \times p}, M_{22} \in \mathbb{C}^{q \times p_1+\dots+p_k}.$$

$$(m_1 \dots m_p \quad m_{p+1} \dots m_{p+q}) \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = (f_0 m_1 \dots f_0 m_p \quad f_1 \cdot \mathcal{B}_1 \dots f_k \cdot \mathcal{B}_k),$$

where  $f_i \cdot \mathcal{B}_i$  is the row vector obtained by multiplying  $f_i$  with the monomials in  $\mathcal{B}_i \cap \mathbb{Z}^n$ ,  $1 \leq i \leq k$ .

**Theorem 2.** *There exists a matrix  $F \in \mathbb{C}^{p_1+\dots+p_k \times p}$  such that every solution  $\xi$  of the sparse system determine a left eigenvector of  $M_{11} + M_{12}F$ . Even more,  $f_0(\xi)$  is the eigenvalue of that left eigenvector.*

*The matrix  $F$  satisfies the linear condition  $M_{21} + M_{22}F = 0$ .*

PROOF. First, let us see that the rank of  $M_{22}$  is  $q$ . The matrix  $M_{22}$  is the matrix of the composition of the following maps,

$$S_{\mathcal{B}_1} \times \dots \times S_{\mathcal{B}_k} \xrightarrow{\Phi} S_{\mathcal{E}} \xrightarrow{\pi} S_{\mathcal{E}}/S_{\mathcal{B}_0},$$

where  $\pi$  is the quotient map. The rank of  $M_{22}$  is equal to  $\text{rk}(\pi \Phi)$ ,

$$\text{im}(\pi \Phi) = \pi(\text{im} \Phi) = \text{im} \Phi / \text{im}(\Phi)_{\mathcal{B}_0}, \quad \text{im}(\Phi)_{\mathcal{B}_0} = \text{im}(\Phi) \cap S_{\mathcal{B}_0}.$$

Let  $t$  be the rank of  $M_{22}$ , and let  $q$  be the dimension of  $S_{\mathcal{E}}/S_{\mathcal{B}_0}$ ,

$$t = \dim(\text{im} \Phi / \text{im}(\Phi)_{\mathcal{B}_0}), \quad q = \dim(S_{\mathcal{E}}/S_{\mathcal{B}_0}) = \dim(S_{\mathcal{E}}) - \dim(S_{\mathcal{B}_0}) = (p+q) - p.$$

Using Theorem 1, we obtain

$$\dim(S_{\mathcal{E}}/\text{im} \Phi) = \dim(S_{\mathcal{B}_0}/\text{im}(\Phi)_{\mathcal{B}_0}) \implies q = t.$$

Now that we know that  $\text{rk}(M_{22}) = q$ , it is easy to prove that there exists a matrix  $F \in \mathbb{C}^{p_1+\dots+p_k \times p}$  such that

$$M_{22}F = -M_{21}.$$

Each column of  $F$ ,  $c_1, \dots, c_p$ , is a solution of the linear system  $M_{22}c_i = b_i$ , where  $b_i \in \mathbb{C}^q$  is the  $i$ -column vector of  $-M_{21}$ ,  $1 \leq i \leq p$ .

Let  $\xi \in \mathbb{C}^n$  be a solution of the sparse system,  $f_1(\xi) = \dots = f_k(\xi) = 0$ . Then

$$\begin{aligned} & \begin{pmatrix} \xi^{m_1} \dots \xi^{m_p} & \xi^{m_{p+1}} \dots \xi^{m_{p+q}} \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = f_0(\xi) \cdot \begin{pmatrix} \xi^{m_1} \dots \xi^{m_p} & 0 \end{pmatrix} \implies \\ & \begin{pmatrix} \xi^{m_1} \dots \xi^{m_p} & \xi^{m_{p+1}} \dots \xi^{m_{p+q}} \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ F & I \end{pmatrix} = f_0(\xi) \cdot \begin{pmatrix} \xi^{m_1} \dots \xi^{m_p} & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ F & I \end{pmatrix} \implies \\ & (\xi^{m_1} \dots \xi^{m_p})(M_{11} + M_{12}F) = f_0(\xi) \cdot (\xi^{m_1} \dots \xi^{m_p}). \end{aligned}$$

Then,  $\xi$  determines a left eigenvector of  $M_{11} + M_{12}F$  with eigenvalue  $f_0(\xi)$ .  $\square$

The following is an algorithm to compute all the solutions of a sparse system,

$$\left\{ \begin{array}{lcl} f_1 = \sum_{v \in \mathcal{A}_1 \cap \mathbb{Z}^n} c_{1,v} x_1^{v_1} \dots x_n^{v_n} & = & 0 \\ & \vdots & \\ f_k = \sum_{v \in \mathcal{A}_k \cap \mathbb{Z}^n} c_{k,v} x_1^{v_1} \dots x_n^{v_n} & = & 0 \end{array} \right.$$

Let  $f_0$  be a non-constant general Laurent polynomial with  $N(f_0) = \mathcal{A}_0$ . Assume that the Newton polytopes  $\mathcal{A}_0, \dots, \mathcal{A}_k$  are normal. In other case, take  $a_i \cdot \mathcal{A}_i$  for some natural number  $a_i \leq n-1$ ,

$0 \leq i \leq k$ . Assume also that  $0 \in \mathcal{A}_0 \cap \dots \cap \mathcal{A}_k$ ,

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**Input:** A sparse system with finitely many solutions and a non-constant Laurent polynomial.  
**Output:** The solutions of the system.

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1. Let  $\mathcal{E} = \mathcal{A}_0 + \dots + \mathcal{A}_k$  and let  $\mathcal{B}_i = \mathcal{A}_0 + \dots + \widehat{\mathcal{A}_i} + \dots + \mathcal{A}_k$ .
2. Get ordered monomial bases, where  $\mathcal{B}_0 \cap \mathbb{Z}^n = \{m_1, \dots, m_p\}$  and  $\mathcal{E} \cap \mathbb{Z}^n = \{m_1, \dots, m_p, m_{p+1}, \dots, m_{p+q}\}$ . Assume  $m_1 = 0$ .
3. Compute the matrix  $M$  and decompose it as  $[M_{11}, M_{12}; M_{21}, M_{22}]$ .
4. Solve the linear system  $M_{22}F = -M_{21}$ .
5. Compute left eigenvectors of  $M_{11} + M_{12}F$ .
6. For each left eigenvector  $v$  such that  $v_1 \neq 0$  do:
  - 6.1 Normalize  $v$  as  $v = (1, v_2, \dots)$ .
  - 6.2 Let  $x$  be such that  $x^{m_i} = v_i$  for all monomial  $m_i \in \mathcal{B}_0 \cap \mathbb{Z}^n$ ,  $1 \leq i \leq p$ .
  - 6.3 Save  $x$  if it is a solution.
7. Return the solutions.

**Remark.** The formalism of Newton polytopes in our method to solve a sparse system is necessary. For example, consider the system in  $S = \mathbb{C}[x_1, x_2]$ ,

$$\begin{cases} 1 - x_1 x_2 &= 0 \\ 1 - x_2 &= 0 \end{cases}$$

Let  $S_d$  be the space of polynomial of degree  $d$ . The support of a generic polynomial in  $S_d$  is

$$\{(a_1, a_2) \in (\mathbb{Z}_{\geq 0})^2 : a_1 + a_2 \leq d\}.$$

Fix a natural number  $e$  and consider the map  $\Phi$ ,

$$\Phi : S_{e-2} \times S_{e-1} \rightarrow S_e, \quad \Phi(g_1, g_2) = g_1 \cdot (1 - x_1 x_2) + g_2 \cdot (1 - x_2).$$

Given that  $\dim(S/\langle 1 - x_1 x_2, 1 - x_2 \rangle) = 1$ , we need that the codimension of  $\text{im}(\Phi)$  be 1. But this never happens because the monomials 1 and  $x_1^e$  are linearly independent from  $\text{im}(\Phi)$ .

## 2. Adaptation I: affine system of trilinear equations.

Let  $S = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_s]$  be the polynomial algebra. Given three non-negatives integers,  $(d_1, d_2, d_3)$  consider the Newton polytope

$$\mathcal{A}_{d_1, d_2, d_3} = (d_1 \cdot \Delta_n) \times (d_2 \cdot \Delta_m) \times (d_3 \cdot \Delta_s),$$

$$\Delta_k = \{t \in (\mathbb{R}_{\geq 0})^k : t_1 + \dots + t_k \leq 1\}, \quad k \in \{n, m, s\}.$$

Let us denote  $S_{d_1, d_2, d_3}$  instead of  $S_{\mathcal{A}_{d_1, d_2, d_3}}$ ,

$$S_{d_1, d_2, d_3} = \{f \in S : N(f) \subseteq \mathcal{A}_{d_1, d_2, d_3}\}.$$

We say that a polynomial in  $S_{d_1, d_2, d_3}$  has *multidegree* less than or equal to  $(d_1, d_2, d_3)$ . For example, the multidegree of  $x_1 x_2 y_1 + z_1^2 z_2$  is less than or equal to  $(2, 1, 3)$ . We say that a polynomial  $\ell$  is *trilinear*, if the multidegree of  $\ell$  is less than or equal to  $(1, 1, 1)$ .

Consider the following affine system of trilinear equations,

$$\begin{cases} \ell_1(x, y, z) = 0 \\ \vdots \\ \ell_k(x, y, z) = 0 \end{cases}, \quad x \in \mathbb{C}^n, y \in \mathbb{C}^m, z \in \mathbb{C}^s.$$

Assume that the system has finitely many solutions, say  $r$ . Let  $f$  be a polynomial such that,  $N(f) \subseteq S_{1,1,1}$ . For simplicity, take  $f$  a generic linear form. Note that  $f, \ell_1, \dots, \ell_k \in S_{1,1,1}$ .

By Theorem 1, the following linear map is surjective,

$$\Psi : (S_{k,k,k})^{k+1} \longrightarrow S_{k+1,k+1,k+1}, \quad \Psi(g, g_1, \dots, g_k) = g \cdot f + \sum_{i=1}^k g_i \cdot \ell_i.$$

Take  $B$  a monomial ordered basis of  $S_{k,k,k}$  and extend it to a monomial ordered basis,  $E$ , of  $S_{k+1,k+1,k+1}$ .

$$B = \mathcal{A}_{k,k,k} \cap \mathbb{Z}^{n+m+s} = \{m_1, \dots, m_p\},$$

$$E = \mathcal{A}_{k+1,k+1,k+1} \cap \mathbb{Z}^{n+m+s} = \{m_1, \dots, m_p, m_{p+1}, \dots, m_{p+q}\},$$

where  $m_i$  is a monomial,  $1 \leq i \leq p+q$ . Note that  $1, x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_s \in B$ . Assume  $m_1 = 1$ .

Let  $M \in \mathbb{C}^{(p+q \times (k+1)p}$  be the rectangular matrix in bases  $B$  and  $E$  of the linear map  $\Psi$ ,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad M_{11} \in \mathbb{C}^{p \times p}, M_{22} \in \mathbb{C}^{q \times kp}.$$

By Theorem 2 there exists a matrix  $F \in \mathbb{C}^{kp \times p}$  such that the left eigenvectors of  $M_{11} + M_{12}F$  determine possible solutions of the system of trilinear equations. Even more, its eigenvalues are the possible values of  $f$  at the solutions of the system.

As a corollary, we get an algorithm to compute solutions of a system of trilinear equations using linear algebra tools.

- 
- |   |
|---|
| <b>Input:</b><br>A system of trilinear equations with<br>finitely many solutions and a generic linear form.<br><b>Output:</b><br>The solutions of the system. |
|---|
- 
1. Construct the bases  $B$  and  $E$ .
  2. Compute the matrix  $M$  and decompose it as  $[M_{11}, M_{12}; M_{21}, M_{22}]$ .
  3. Solve the linear system  $M_{22}F = -M_{21}$ .
  4. Find left eigenvectors of  $M_{11} + M_{12}F$ .
  5. For each left eigenvector  $v$  such that  $v_1 \neq 0$  do:
    - 5.1 Normalize  $v$  as  $v = (1, v_2, \dots)$ .
    - 5.2 Let  $x = (x_1, \dots, x_n)$  be such that  $x_i = v_{\sigma_i}$  where  $\sigma_i$  is the coordinate of  $x_i$  in  $B$ ,  $1 \leq i \leq n$ .
    - 5.3 Same for  $y$  and  $z$ .
    - 5.4 Save  $(x, y, z)$  if it is a solution.
  6. Return the solutions.

### 3. Adaptation II: affine system of polynomial equations.

Consider the following affine system of polynomial equations in  $S = \mathbb{C}[x_1, \dots, x_n]$ ,

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_k(x_1, \dots, x_n) = 0 \end{cases}$$

where the support of  $f_i$  is  $\{(a_1, \dots, a_n) \in (\mathbb{Z}_{\geq 0})^n : a_1 + \dots + a_n \leq d_i\}$ ,  $1 \leq i \leq k$ . In particular,  $f_i$  is a polynomial of degree  $d_i$ ,  $\deg(f_i) = d_i$ ,  $1 \leq i \leq k$ . Assume that the system has finitely many solutions, say  $r$ . Let  $f$  be a generic polynomial of degree  $d$ .

By Theorem 1, there exist a degree  $e = d + d_1 + \dots + d_k$  such that the following linear map is surjective,

$$\Psi : S_{e-d} \times S_{e-d_1} \times \dots \times S_{e-d_k} \longrightarrow S_e, \quad \Psi(g, g_1, \dots, g_k) = g \cdot f + \sum_{i=1}^k g_i \cdot f_i.$$

Choose monomial ordered bases of  $S_{e-d}, S_{e-d_1}, \dots, S_{e-d_k}$ , and extend them to a basis of  $S_e$ . Let us denote  $B_0$  the basis of  $S_{e-d}$ ,  $B_i$  the basis of  $S_{e-d_i}$ ,  $1 \leq i \leq k$ , and  $E$  the basis of  $S_e$ . Let  $p$  be the cardinal of  $B_0$ ,  $p_i$  the cardinal of  $B_i$ ,  $1 \leq i \leq k$  and  $p+q$  be the cardinal of  $E$ ,

$$B_0 = \{m_1, \dots, m_p\}, \quad E = \{m_1, \dots, m_p, m_{p+1}, \dots, m_{p+q}\},$$

where  $m_i$  is a monomial,  $1 \leq i \leq p+q$ . Note that  $1, x_1, \dots, x_n \in B_0$ . Assume  $m_1 = 1$ .

Let  $M \in \mathbb{C}^{p+q \times p+p_1+\dots+p_k}$  be the matrix assigned to  $\Psi$ ,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad M_{11} \in \mathbb{C}^{p \times p}, M_{22} \in \mathbb{C}^{q \times p_1+\dots+p_k}.$$

By Theorem 2 there exists a matrix  $F \in \mathbb{C}^{p_1+\dots+p_k \times p}$  such that the left eigenvectors of  $M_{11} + M_{12}F$  determine possible solutions of the system of polynomial equations. Even more, its eigenvalues are the possible values of  $f$  at the solutions of the system.

<b>Input:</b>	A system of polynomial equations with $r$ solutions and a generic polynomial of degree $d$ .
<b>Output:</b>	The solutions of the system.
<hr/>	
1.	Construct the bases $B_0, \dots, B_k, E$ .
2.	Compute the matrix $M$ and decompose it as $[M_{11}, M_{12}; M_{21}, M_{22}]$ .
3.	Solve the linear system $M_{22}F = -M_{21}$ .
4.	Find left eigenvectors of $M_{11} + M_{12}F$ .
5.	For each left eigenvector $v$ such that $v_1 \neq 0$ do: <ol style="list-style-type: none"> <li>5.1 Normalize <math>v</math> as <math>v = (1, v_2, \dots)</math>.</li> <li>5.2 Let <math>x = (x_1, \dots, x_n)</math> be such that <math>x_i = v_{\sigma_i}</math> where <math>\sigma_i</math> is the coordinate of <math>x_i</math> in <math>B_0</math>, <math>1 \leq i \leq n</math>.</li> <li>5.3 Save <math>x</math> if it is a solution.</li> </ol>
6.	Return the solutions.

Note that Step 1 of the algorithm has exponential complexity,  $O(n^e)$ . The other steps have polynomial complexity.

#### 4. Identifying the solutions.

Let  $M \in \mathbb{C}^{N \times N}$  be a diagonalizable matrix, let  $\lambda \in \mathbb{C}$  be an eigenvalue and let  $v \in \mathbb{C}^N$  be an eigenvector such that  $Mv = \lambda v$ . In practice, there exists a positive real number  $\varepsilon$  such that

$$\|v - \tilde{v}\| < \varepsilon, \quad |\lambda - \tilde{\lambda}| < \varepsilon,$$

where  $\tilde{v}$  is the computed eigenvector corresponding to  $v$  and  $\tilde{\lambda}$  is the computed eigenvalue corresponding to  $\lambda$ .

Let  $\tilde{x} \in \mathbb{C}^n$  be an estimated solution of the sparse system,  $f_1 = \dots = f_k = 0$ , obtained with our algorithm using the Laurent polynomial  $f_0$ . Let  $x \in \mathbb{C}^n$  be the exact solution corresponding to  $\tilde{x}$ . Then, there exists a constant  $K$  depending on the coefficients of  $f_0, \dots, f_k$  such that,

$$|f_i(\tilde{x})| = |f_i(\tilde{x}) - f_i(x)| \leq K\|\tilde{x} - x\| \leq K\|\tilde{v} - v\| \leq K\varepsilon \leq (K+1)\varepsilon, \quad 1 \leq i \leq k.$$

$$|f_0(\tilde{x}) - \tilde{\lambda}| \leq |f_0(\tilde{x}) - \lambda| + |\lambda - \tilde{\lambda}| \leq |f_0(\tilde{x}) - f_0(x)| + \varepsilon \leq K\|\tilde{v} - v\| + \varepsilon \leq (K+1)\varepsilon.$$

Finally, we will identify  $\tilde{x}$  as a solution of the sparse system if  $|f_i(\tilde{x})| < (K+1)\varepsilon$ ,  $1 \leq i \leq k$  and  $|f_0(\tilde{x}) - \tilde{\lambda}| < (K+1)\varepsilon$ . The constant  $K$  may be taken as the sum of the magnitudes of the coefficients of the polynomials. In other words,  $K = \|M\|_1$ , where  $M$  is the matrix assigned to  $\Psi$ .

## 5. Application.

To conclude this article, let us give an application to compute the maximum of a generic trilinear form over a product of spheres,

$$\ell : \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \times \mathbb{R}^{s+1} \rightarrow \mathbb{R}, \quad \ell(x, y, z) = \sum_{(i,j,k)=0}^{(n,m,s)} a_{ijk} x_i y_j z_k, \quad \max_{\|x\|=\|y\|=\|z\|=1} |\ell(x, y, z)|,$$

where the norm is the usual 2-norm. In the literature, this maximum is called the first singular value of  $\ell$ , [10, 3].

Using Lagrange method of multipliers, [1, §13.7], the extreme points of  $\ell$  over a product of spheres,  $\mathbb{S}^n \times \mathbb{S}^m \times \mathbb{S}^s$ , satisfy

$$\begin{cases} \partial\ell/\partial x_i(x_0, \dots, x_n, y_0, \dots, y_m, z_0, \dots, z_s) = 2\alpha x_i, & 0 \leq i \leq n, \\ \partial\ell/\partial y_j(x_0, \dots, x_n, y_0, \dots, y_m, z_0, \dots, z_s) = 2\beta y_j, & 0 \leq j \leq m, \\ \partial\ell/\partial z_k(x_0, \dots, x_n, y_0, \dots, y_m, z_0, \dots, z_s) = 2\lambda z_k, & 0 \leq k \leq s, \end{cases}$$

$$\alpha, \beta, \lambda \in \mathbb{R}, \quad \|x\| = \|y\| = \|z\| = 1.$$

These equations imply that the vector  $\partial\ell/\partial x(x, y, z)$  is a multiple of  $x$ . Same for  $y$  and  $z$ . In other words, considering the system in  $\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^s$ , we can hide the variables  $\alpha, \beta$  and  $\lambda$ ,

$$\begin{cases} x_j \partial\ell/\partial x_i(x, y, z) = x_i \partial\ell/\partial x_j(x, y, z), & 0 \leq i < j \leq n, \\ y_j \partial\ell/\partial y_i(x, y, z) = y_i \partial\ell/\partial y_j(x, y, z), & 0 \leq i < j \leq m, \\ z_j \partial\ell/\partial z_i(x, y, z) = z_i \partial\ell/\partial z_j(x, y, z), & 0 \leq i < j \leq s. \end{cases}$$

Given that  $\ell$  is trilinear, the expression  $x_j \partial\ell/\partial x_i(x, y, z)$  is equal to  $\ell(x_j e_i, y, z)$ , where  $e_i \in \mathbb{R}^{n+1}$  is the vector with 1 in the  $i$ -coordinate and 0 in the rest. Same for  $y$  and  $z$ . Summing up, the extreme points of  $\ell$  satisfy the following system of equations in  $\mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \times \mathbb{R}^{s+1}$ ,

$$\begin{cases} \ell(x_j e_i - x_i e_j, y, z) = 0, & 0 \leq i < j \leq n, \\ \ell(x, y_j e_i - y_i e_j, z) = 0, & 0 \leq i < j \leq m, \\ \ell(x, y, z_j e_i - z_i e_j) = 0, & 0 \leq i < j \leq s, \\ x_0^2 + \dots + x_n^2 = 1 \\ y_0^2 + \dots + y_m^2 = 1 \\ z_0^2 + \dots + z_s^2 = 1 \end{cases}$$

If  $\ell$  is generic, we may assume that the extreme points are finite. Note that we have more equations than variables. Enumerate the equations,  $f_1, \dots, f_k$ , and let  $r$  be the number of solutions. We are going to apply our method to the system of polynomial equations  $f_1, \dots, f_k$ , and to  $\ell$ . Let  $\lambda_1, \dots, \lambda_r$  be eigenvalues of  $M_{11} + M_{12}F$  assigned to the solutions of the system  $f_1 = \dots = f_k = 0$  and to the generic trilinear form  $\ell$ . If  $|\lambda_1| \geq |\lambda_i|$ ,  $2 \leq i \leq r$ , then  $|\lambda_1|$  is the maximum value of  $\ell$  over  $S^n \times S^m \times S^s$ .

The following examples were computed using the algorithm programmed in Octave, see Appendix A for a similar code.

**Example.** Let  $\ell : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be the following trilinear form

$$\ell(x, y, z) = 4x_1y_1z_1 + x_2y_1z_1 - 5x_1y_2z_1 - 5z_1x_2y_2 + 2x_1y_1z_2 - 7y_1x_2z_2 - 9x_1y_2z_2 - 6x_2y_2z_2.$$

Then the maximum is,

$$\max_{\|x\|=\|y\|=\|z\|=1} |\ell(x, y, z)| = 12.87128226.$$

**Example.** Let  $\ell : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be the following trilinear form

$$\begin{aligned} \ell(x, y, z) = & 4x_1y_1z_1 + x_2y_1z_1 + 6x_3y_1z_1 - 5x_1y_2z_1 - 5z_1y_2x_2 - 6z_1y_2x_3 + \\ & 2x_1y_1z_2 - 7x_2y_1z_2 - 3x_3y_1z_2 - 9y_2z_2x_1 - 6y_2z_2x_2 - 9y_2z_2x_3. \end{aligned}$$

Then the maximum is,

$$\max_{\|x\|=\|y\|=\|z\|=1} |\ell(x, y, z)| = 16.81951586.$$

**Example.** Let  $\ell : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$  be the following trilinear form

$$\begin{aligned} \ell(x, y, z) = & -x_3y_1z_3 + 4x_1y_1z_1 + 3x_3y_2z_1 - 6x_1y_3z_2 - 3y_1z_1x_2 + 4x_3y_1z_1 - 9x_1z_1y_2 + \\ & 7z_1x_2y_2 - 5x_1y_3z_1 + 8x_2y_3z_1 + 2x_3y_3z_1 + 2x_1y_1z_2 - 6y_1z_2x_2 - 3x_3y_1z_2 + \\ & 9z_2x_2y_2 + 3x_2y_3z_2 + 5x_3y_3z_2 - 5x_1y_1z_3 - 9y_1z_3x_2 - 7x_1z_3y_2 - 2x_3y_2z_3 + 6x_1y_3z_3 + \\ & 7x_2y_3z_3 - 10x_3y_3z_3 + x_1z_2y_2 + x_3y_2z_2. \end{aligned}$$

Then the maximum is,

$$\max_{\|x\|=\|y\|=\|z\|=1} |\ell(x, y, z)| = 19.57534001.$$

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### A. Octave code.

Let us write a Octave code to solve a system of two polynomials in two variables. First of all we will identify a polynomial  $f$  with the following data structure,

$$f(x_1, \dots, x_n) = \sum_{w \in \mathcal{A} \cap \mathbb{Z}^n} c_w x_1^{w_1} \dots x_n^{w_n}, \quad \mathcal{A} \cap \mathbb{Z}^n = \{v_1, \dots, v_p\},$$

$$f.mon\{i\} = v_i, \quad f.coef\{i\} = c_{v_i}, \quad 1 \leq i \leq p, \quad f.terms = p.$$

It is easy to write a program that prints a polynomial,

```
function s=printPoly(f)
s="";
for i=1:f.terms
    v=f.mon\{i\};
    n=size(v,2);
    l=f.coef\{i\};
    if(i==1||l<0)
        s=strcat(s,num2str(l));
    else
        s=strcat(s,"+",num2str(l));
    end
    for j=1:n
        s=strcat(s,"*x[",num2str(j),"]^",num2str(v(j)));
    end
end;
end
```

Another useful program is the evaluation of  $f$  at a point  $(x_1, \dots, x_n)$ ,

```
function y=evalPoly(f,x)
y=0;
for i=1:f.terms
    v=f.mon\{i\};
    l=f.coef\{i\};
    y+=l*prod(x.^v);
end;
end
```

Finally, here is the Octave code that solves a system of two polynomials in two variables using  $f_0$  a linear form,  $f_1$  a polynomial of degree 5 and  $f_2$  of degree 7. The reader may change the values.

```
d=[1,5,7];
e=sum(d);
%Generate random polynomials of degree d_k
for k=1:size(d,2)
    cont=1;
    for i=0:d(k)
        for j=0:d(k)
            if i+j<=d(k)
                f{k}.mon\{cont\}=[i,j];
                f{k}.coef\{cont\}=round(rand*20-10);
                cont++;
            end
        end;
    end;
    f{k}.terms=cont-1;
```

```

end;
%Generate monomial bases of S_{e-d_k}
for k=1:size(d,2)
    cont=1;
    for i=0:e-d(k)
        for j=0:e-d(k)
            if i+j<=e-d(k)
                mons{k}.pos{cont}=[i,j];
                cont++;
            end
        end;
    end;
    mons{k}.siz=cont-1;
end;
%Generate monomial basis of S_e appending the monomials from S_{e-d_1}.
%For technical reasons, the monomials are shifted by one.
monE=zeros(e+1,e+1);
for i=1:mons{1}.siz
    v=mons{1}.pos{i};
    monE(v(1)+1,v(2)+1)=i;
end;
cont=mons{1}.siz+1;
for i=0:e
    for j=0:e
        if (i+j<=e && monE(i+1,j+1)==0)
            monE(i+1,j+1)=cont;
            cont++;
        end
    end;
end;
monE_siz=cont-1;
%Generate Matrix M
M=[];
for i=1:size(d,2)
    for j=1:mons{i}.siz
        col=zeros(monE_siz,1);
        for k=1:f{i}.terms
            v=mons{i}.pos{j}+f{i}.mon{k};
            col(monE(v(1)+1,v(2)+1))=f{i}.coef{k};
        end
        M=[M,col];
    end
end
%Generate Matrix decomposition,
M11=M(1:mons{1}.siz,1:mons{1}.siz);
M12=M(1:mons{1}.siz,mons{1}.siz+1:size(M,2));
M21=M(mons{1}.siz+1:size(M,1),1:mons{1}.siz);
M22=M(mons{1}.siz+1:size(M,1),mons{1}.siz+1:size(M,2));
%Compute left eigenvectors
F=M22\(-M21);
res=M11+M12*F;
[v,1]=eig(res');
%Find solutions
for k=1:size(v,2)
    if abs(v(1,k))>0
        posx1=monE(2,1);
        posx2=monE(1,2);
        x=[v(posx1,k),v(posx2,k)]/v(1,k);
        if norm([evalPoly(f{1},x)-l(k,k),evalPoly(f{2},x),evalPoly(f{3},x)])<1

```

```

x
end
end
end

```

The following table shows the time in seconds used to solve Step 1 throw 4 of the algorithm. In the first column appears the degree of the polynomials, in the second the time of our method and in the third, the time used with a Gröbner Basis algorithm. All the computations were made on a 2.1GHz CPU, with 2GB of memory. We used part of the previous code to generate the second column and we used the following Maple 11 code to generate the third column,

```

> G:=Basis([f[2],f[3]],'tord'):
> ns,rv:=NormalSet(G, tord):
> mulMat:=MultiplicationMatrix(f[1],ns,rv,G,tord):
> evalf(eigenvectors(evalf(mulMat))):

```

The resulting table is the following,

<i>degrees</i>	Steps 1-4	Gröbner
(1, 5, 7)	0.24	2.56
(1, 5, 9)	0.37	4.91
(1, 5, 11)	0.56	9.77
(1, 7, 9)	0.62	14.46
(1, 7, 11)	0.89	25.19
(1, 9, 11)	1.34	57.67
(1, 11, 11)	2.04	114.24

## References.

### References

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